Fourier Analysis

Mar 31, 2022

Review.

- Thm (Fourier inversion formula)

  Let $f \in \mathcal{M}(\mathbb{R})$. Suppose $\hat{f} \in \mathcal{M}(\mathbb{R})$.

  Then

$$f(x) = \int_{\mathbb{R}} \hat{f}(\xi) e^{2\pi i \xi x} d\xi$$

- Thm (Plancherel formula).

  Let $f \in \mathcal{M}(\mathbb{R})$. Suppose that $\hat{f} \in \mathcal{M}(\mathbb{R})$.

  Then

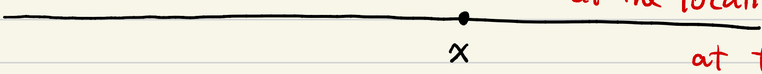
$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 d\xi.$$

- Schwartz space $S(\mathbb{R})$.
  - $f \in S(\mathbb{R})$ if $f \in C^{\infty}(\mathbb{R})$ and

    $\sup_{x \in \mathbb{R}} |x^n f^{(\ell)}(x)| < \infty$ for all $n, \ell \geq 0$.
  - $f \in S(\mathbb{R}) \Leftrightarrow \hat{f} \in S(\mathbb{R})$.

§ 5.2 Application (I): Heat equation on $\mathbb{R}$.

$u(x,t)$ — temperature at the location x at time t.



$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, & x \in \mathbb{R},\ t > 0 \quad ① \\ u(x,0) = f(x), & x \in \mathbb{R} \quad ② \end{cases}$$

(initial condition)

Let us use the method of Fourier transform to derive a solution for the above heat equation.

Taking Fourier transform on the both sides of ① (in variable x),

we obtain

$$\frac{\partial}{\partial t} \hat{u}(\xi, t) = (2\pi i \xi)^2 \, \hat{u}(\xi, t).$$

Justification:

Set $\hat{u}(\xi,t) = \int_{\mathbb{R}} u(x,t)\, e^{-2\pi i \xi x}\, dx$

Then

$$\int_{\mathbb{R}} \frac{\partial u}{\partial t}(x,t)\, e^{-2\pi i \xi x}\, dx$$

$$= \frac{\partial}{\partial t} \int_{\mathbb{R}} u(x,t)\, e^{-2\pi i \xi x}\, dx$$

$$= \frac{\partial}{\partial t} \hat{u}(\xi,t).$$

$$\int_{\mathbb{R}} \frac{\partial^2 u(x,t)}{\partial x^2} \cdot e^{-2\pi i \xi x}\, dx$$

$$= (2\pi i \xi)^2\, \hat{u}(\xi,t).$$

Hence

$$\frac{d\,\hat{u}(\xi,t)}{dt} = -4\pi^2\xi^2\,\hat{u}(\xi,t)$$

So

$$\frac{d\,\hat{u}(\xi,t)}{dt} + 4\pi^2\xi^2\,\hat{u}(\xi,t) = 0.$$

$$e^{4\pi^2\xi^2 t}\cdot\left(\frac{d\,\hat{u}(\xi,t)}{dt} + 4\pi^2\xi^2\,\hat{u}(\xi,t)\right) = 0$$

So $$\frac{d}{dt}\left(e^{4\pi^2\xi^2 t}\,\hat{u}(\xi,t)\right) = 0.$$

$$e^{4\pi^2\xi^2 t}\,\hat{u}(\xi,t) = A(\xi).$$

$$\Rightarrow \quad \hat{u}(\xi,t) = A(\xi)\cdot e^{-4\pi^2\xi^2 t}.$$

Taking $t=0$ in the above equation gives

$$\hat{u}(\xi,0) = A(\xi)$$

Recall that $u(x,0) = f(x)$. It follows that

$$\hat{u}(\xi,0) = \hat{f}(\xi).$$

Hence $A(\xi) = \hat{f}(\xi)$. Therefore

$$\hat{u}(\xi,t) = \hat{f}(\xi) \cdot e^{-4\pi^2\xi^2 t}.$$

Notice that $e^{-4\pi^2\xi^2 t}$ is the Fourier transform of

$$\mathcal{H}_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}, \quad x \in \mathbb{R}, \ t > 0.$$

We call $\{\mathcal{H}_t\}_{t>0}$ the heat kernel on $\mathbb{R}$ as $t \to 0$.

So

$$\widehat{u}(\xi, t) = \hat{f}(\xi) \cdot \widehat{\mathcal{H}_t}(\xi)$$

$$= \widehat{f * \mathcal{H}_t}(\xi)$$

By the Fourier inversion formula, we obtain

$$u(x,t) = f * \mathcal{H}_t(x).$$

Next we give a theoretic check.

Thm. Let $f \in S(\mathbb{R})$. Let $u(x,t) = f * \mathcal{H}_t(x)$.
Then the following holds:

① $u \in C^\infty(\mathbb{R} \times \mathbb{R}_+)$, where $\mathbb{R}_+ := (0, \infty)$,
$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$.

② $u(x,t) \rightrightarrows f(x)$ as $t \to 0$

③ $\int_{\mathbb{R}} |u(x,t) - f(x)|^2 dx \to 0$ as $t \to 0$

Pf: ① Since $f, \mathcal{H}_t \in S(\mathbb{R})$,

So are $\hat{f}$ and $\hat{\mathcal{H}}_t$. It follows that

$\hat{f} \cdot \hat{\mathcal{H}}_t \in S(\mathbb{R})$. That is, $\widehat{f * \mathcal{H}_t} \in S(\mathbb{R})$.

Hence

$$f * \mathcal{H}_t \in S(\mathbb{R}).$$

So by the inversion formula,

$$u(x,t) = \int_{\mathbb{R}} \widehat{f * \mathcal{H}_t}(\xi) \cdot e^{2\pi i \xi x} \, d\xi$$

$$= \int_{\mathbb{R}} \hat{f}(\xi) \cdot e^{-4\pi^2 \xi^2 t} e^{2\pi i \xi x} \, d\xi$$

In the following, we show $\frac{\partial u}{\partial t}$ exists

$$\frac{\partial u}{\partial t}(x,t) := \lim_{\Delta t \to 0} \frac{u(x, t+\Delta t) - u(x,t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \int_{\mathbb{R}} \hat{f}(\xi) \frac{e^{-4\pi^2 \xi^2 (t+\Delta t)} - e^{-4\pi^2 \xi^2 t}}{\Delta t} \cdot e^{2\pi i \xi x} \, d\xi$$

$$= \lim_{\Delta t \to 0} \int_{\mathbb{R}} \hat{f}(\xi) \cdot e^{-4\pi^2\xi^2 t} \cdot \frac{e^{-4\pi^2\xi^2 \Delta t} - 1}{\Delta t} \cdot e^{2\pi i \xi x} \, d\xi$$

Notice that

$$\left| \frac{e^{-4\pi^2\xi^2 \Delta t} - 1}{\Delta t} \right| \leqslant \text{const} \cdot |\xi^2|$$

(using the mean value theorem)

So

$$\left| \hat{f}(\xi) e^{-4\pi^2\xi^2 t} \cdot \frac{e^{-4\pi^2\xi^2 \Delta t} - 1}{\Delta t} \cdot e^{2\pi i \xi x} \right|$$

$$\lesssim \left| \hat{f}(\xi) \right| \cdot |\xi|^2$$

(Notice that $\hat{f}(\xi) \cdot \xi^2 \in S(\mathbb{R})$)

By the dominated convergence thm,

$$\frac{\partial u(x,t)}{\partial t} = \int_{\mathbb{R}} \hat{f}(\xi) \cdot e^{-4\pi^2\xi^2 t} (-4\pi^2\xi^2) \cdot e^{2\pi i \xi x} \, d\xi$$

In a similar way, we can show $u \in C^\infty(\mathbb{R} \times \mathbb{R}_+)$.

Moreover,

$$\frac{\partial^2 u}{\partial x^2} = \int_{\mathbb{R}} \hat{f}(\xi) \cdot e^{-4\pi^2\xi^2 t} \frac{\partial^2}{\partial x^2}\left(e^{2\pi i \xi x}\right) d\xi$$

$$= \int_{\mathbb{R}} \hat{f}(\xi)\, e^{-4\pi^2\xi^2 t} \cdot \left(-4\pi^2\xi^2\right) \cdot e^{2\pi i \xi x} d\xi.$$

$$= \frac{\partial u}{\partial t}$$

This proves ①.